INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 6, 883–894 (1986)

SOLUTION TO THE ORR–SOMMERFELD EQUATION FOR LIQUID FILM FLOWING DOWN AN INCLINED PLANE: AN OPTIMAL APPROACH

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SUMMARY

The Orr–Sommerfeld equation is solved numerically for a layer of liquid film flowing down an inclined plane under the action of gravity using the sequential gradient-restoration algorithm (SGRA). The method consists of solving the governing equation as it is a Bolza problem in the calculus of variations. The neutral stability curves, eigenvalues and eigenfunctions to the stability problem can be determined simultaneously during the process.

KEY WORDS Convective Instability Orr–Sommerfeld Equation Optimal Control

INTRODUCTION

The Orr–Sommerfeld equation governs the linear stability characteristics of a basic two-dimensional incompressible laminar parallel flow with respect to an infinitesimal two-dimensional disturbance. It is a fourth-order linear homogeneous ordinary differential equation which contains three parameters, first derived independently by Orr[1] and Sommerfeld.[2] The difficulties associated with solving this eigenvalue problem are well known. No exact solution of this equation has been obtained for a general velocity profile, except for a constant plane Poiseuille flow.

Numerous methods have been presented to calculate the eigenvalues and eigenfunctions of the Orr–Sommerfeld equation. An excellent review of various numerical methods for the solution of the equation has been given by Gersting and Jankowski.[3] The Orr–Sommerfeld equation has frequently been applied for investigating the stability of channel flow, Blasius boundary layer profile, shear layer, laminar jet and developed wake. Approximate solutions to the Orr–Sommerfeld equation for free surface flows have also received much attention in the past. The problem of the stability of a layer of liquid flowing down an inclined plane under the action of gravity was formulated rigorously by Yih[4] as an eigenvalue problem for the determination of the complex phase velocity. Benjamin[5] approximated the eigenfunction by using a power series expansion in the co-ordinate normal to the inclined plane and obtained accurate results. Later, Yih,[6] solved the eigenvalue problem by an expansion in powers of the wave number. The expansions used by Benjamin and Yih gave the same results to the flow under consideration if one considers their choice of the reference velocity. The critical Reynolds number Re_c was found to be

$$Re_c = \tfrac{5}{6}\cot\beta. \tag{1}$$

The foregoing analysis is valid for long waves only, that is the assumption of small wave number α has been made in the calculation. Without making such an assumption, Graef[7] performed a numerical calculation by power series expansion of αRe. Whitaker[8] obtained a solution by direct numerical integration, but the method fails to converge in a region near the neutral stability curve.

0271–2091/86/120883–12$06.00

Received 27 January 1986
Revised 22 April 1986

By replacing the velocity with its constant value at the free surface while keeping the second derivative of the velocity at its true value, Anshus and Goren[9] were able to obtain a simpler solution to the equation, and the eigenvalues were able to be determined by a numerical technique. Benney[10] found an asymptotic solution to the problem. Lin[11] employed a method consisting of obtaining asymptotic series solutions of the Orr–Sommerfeld equation by solving the inviscid equation and a related differential equation by use of the Frobenius method. de Bruin[12] solved the same problem by employing the orthonormalization technique. Although all the techniques mentioned previously are reliable, it is felt that a less tedious and more economical method such as the sequential gradient-restoration algorithm (SGRA) could be employed to solve the Orr–Sommerfeld equation. Our principal aim is to present a reliable method for solving Orr–Sommerfeld problems which is simple to understand and to use; at the same time, we wish to obtain good accuracy. It is the purpose of the present paper to examine the stability of liquid flowing down an inclined plane by the SGRA developed by Miele and his associates.[13–17] It is possible to calculate very accurately both the eigenvalues and the eigenfunctions. Some numerical results are presented and compared with the results of other methods.

Following an outline of the physical problem and brief description of the algorithm, the Orr–Sommerfeld equation is solved as it is a Bolza problem in the calculus of variations. Numerical results are presented and compared. Finally, some concluding remarks close the paper.

THE ORR–SOMMERFELD PROBLEM

An incompressible liquid film of thickness d with constant physical properties flows down an inclined plane with an angle β to the horizontal, the gravitational acceleration is balanced by the viscosity force of the fluid. In a Cartesian co-ordinate system (x, y, z), the primary flow is parallel to the x-axis, and the y-axis is normal to the plane directed downward; the origin is taken on the undisturbed free surface. The first rigorous formulation of the problem based on linear stability theory was given by Yih.[4,6] The governing differential equation is the well-known Orr–Sommerfeld equation:

$$\Phi'''' - 2\alpha^2\Phi'' + \alpha^4\Phi = \mathrm{i}\alpha Re[(U-c)(\Phi''-\alpha^2\Phi) - U''\Phi], \tag{2}$$

where U is the steady state velocity distribution, which is given by

$$U = 1{\cdot}5(1-y^2). \tag{3}$$

The boundary conditions corresponding to equation (2) are

wall, $y = 1$:

$$\Phi = 0, \tag{4}$$

$$\Phi' = 0; \tag{5}$$

free stream, $y = 0$:

$$\Phi'' + \left(\alpha^2 - \frac{3}{c-1{\cdot}5}\right)\Phi = 0, \tag{6}$$

$$\frac{\alpha(3\cot\beta + \alpha^2 S_t Re)}{c-1{\cdot}5}\Phi + \alpha[Re(c-1{\cdot}5) + 3\alpha\mathrm{i}]\Phi' - \mathrm{i}\Phi''' = 0, \tag{7}$$

in which the prime denotes a derivative with respect to y. The boundary conditions (4) and (5) are the no-slip conditions at the wall. By neglecting the viscosity of the air above the liquid film, equation (6) expresses the fact that there can be no tangential stress at the free surface. The last boundary condition (7) shows a balance of the normal stress and the surface tension at the surface.

Equation (2), together with the linearized boundary conditions (4)–(7) constitutes an eigenvalue problem. A non-trivial function Φ satisfies all these conditions only if there exists a functional relationship such that

$$F(\alpha, Re, c, S_t, \beta) = 0.$$

Our main objective is to seek a solution of the form $c = c(\alpha, Re)$ for given values of β and S_t such that the eigenvalue relation is satisfied. In general, the eigenvalue c is a complex quantity, i.e.

$$c = c_r(\alpha, Re) + ic_i(\alpha, Re).$$

The sign of the imaginary part, c_i, determines the stability condition for a given flow. For a real α, if $c_i > 0$, the disturbance remains of constant amplitude in the x-direction but grows exponentially in time; therefore the flow is unstable according to linear theory. If $c_i < 0$, the disturbance will decay in time, and the flow will be stable with respect to infinitesimal disturbances. Thus the α-Re plane may be divided into regions where $c_i < 0$ and regions where $c_i > 0$. These regions are separated by the curve denoted by $c_i(\alpha, Re) = 0$. This curve is the neutral stability curve on which infinitesimal waves are neither damped nor amplified. The lower extremum of the neutral stability curve defines a minimum Reynolds number, the critical Reynolds number, Re_c, at which instability will set in. A primary objective of hydrodynamic stability analysis is the determination of this critical value.

METHOD OF SOLUTION

The method employed in this study is to solve the governing equation (2) as it is a Bolza problem. In this section we outline the Bolza problem in classical optimal control theory and introduce the sequential gradient-restoration algorithm (SGRA) which is useful for solving the resulting differential system. It is then followed by the application of the SGRA to the stability problem.

Bolza problem

Minimize the functional

$$I = \int_0^1 f(\boldsymbol{\chi}, \mathbf{u}, \boldsymbol{\pi}, y)\,dy + [h(\boldsymbol{\chi}, \boldsymbol{\pi})]_0 + [g(\boldsymbol{\chi}, \boldsymbol{\pi})]_1, \tag{8}$$

with respect to the state $\boldsymbol{\chi}(y)$, the control $\mathbf{u}(y)$, and the parameter $\boldsymbol{\pi}$ which satisfy the following constraints:

$$\boldsymbol{\chi}' - \boldsymbol{\phi}(\boldsymbol{\chi}, \mathbf{u}, \boldsymbol{\pi}, y) = 0, \quad 0 \leqslant y \leqslant 1, \tag{9a}$$

$$\mathbf{S}(\boldsymbol{\chi}, \mathbf{u}, \boldsymbol{\pi}, y) = 0, \quad 0 \leqslant y \leqslant 1, \tag{9b}$$

$$[\boldsymbol{\omega}(\boldsymbol{\chi}, \boldsymbol{\pi})]_0 = 0, \tag{9c}$$

$$[\boldsymbol{\psi}(\boldsymbol{\chi}, \boldsymbol{\pi})]_1 = 0. \tag{9d}$$

In the above equations, the functions f, h, g are scalar, and the functions $\boldsymbol{\phi}, \mathbf{S}, \boldsymbol{\omega}, \boldsymbol{\psi}$ are vectors of appropriate dimensions. The independent variable y is a scalar, and the dependent variables $\boldsymbol{\chi}, \mathbf{u}, \boldsymbol{\pi}$, are vectors of appropriate dimensions. The subscript 0 denotes the initial point, and the subscript 1 denotes the final point.

Let $\boldsymbol{\lambda}(y)$, $\boldsymbol{\rho}(y)$, $\boldsymbol{\sigma}$, $\boldsymbol{\mu}$ denote Lagrange multipliers associated with the constraints (9). With this understanding, the first-order optimality conditions take the form

$$\boldsymbol{\lambda}' - f_\chi + \boldsymbol{\phi}_\chi \boldsymbol{\lambda} - \mathbf{S}_\chi \boldsymbol{\rho} = \mathbf{0}, \quad 0 \leqslant y \leqslant 1, \tag{10a}$$

$$f_u - \boldsymbol{\phi}_u \boldsymbol{\lambda} + \mathbf{S}_u \boldsymbol{\rho} = \mathbf{0}, \quad 0 \leqslant y \leqslant 1, \tag{10b}$$

$$\int_0^1 (f_\pi - \boldsymbol{\phi}_\pi \boldsymbol{\lambda} + \mathbf{S}_\pi \boldsymbol{\rho}) \mathrm{d}y + (h_\pi + \boldsymbol{\omega}_\pi \boldsymbol{\sigma})_0 + (g_\pi + \boldsymbol{\psi}_\pi \boldsymbol{\mu})_1 = \mathbf{0}, \tag{10c}$$

$$(-\boldsymbol{\lambda} + h_\chi + \boldsymbol{\omega}_\chi \boldsymbol{\sigma})_0 = \mathbf{0}, \tag{10d}$$

$$(\boldsymbol{\lambda} + g_\chi + \boldsymbol{\psi}_\chi \boldsymbol{\mu})_1 = \mathbf{0}. \tag{10e}$$

Summarizing, we seek the functions $\boldsymbol{\chi}(y)$, $\mathbf{u}(y)$, $\boldsymbol{\pi}$ and the multipliers $\boldsymbol{\lambda}(y)$, $\boldsymbol{\rho}(y)$, $\boldsymbol{\sigma}, \boldsymbol{\mu}$ such that the feasibility equations (9) and the optimality conditions (10) are satisfied.

Approximation method

In general, the differential system (9)–(10) is non-linear, and an approximation method must be used to seek a solution iteratively. In this connection, let P denote the norm squared of the errors associated with the feasibility equation (9), and let Q denote the norm squared of the errors associated with the optimality condition (10):

$$P = \int_0^1 N(\boldsymbol{\chi}' - \boldsymbol{\phi}) \mathrm{d}y + \int_0^1 N(\mathbf{S}) \mathrm{d}y + N(\boldsymbol{\omega})_0 + N(\boldsymbol{\psi})_1, \tag{11}$$

$$\begin{aligned} Q = {} & \int_0^1 N(\boldsymbol{\lambda}' - f_\chi + \phi_\chi \boldsymbol{\lambda} - S_\chi \boldsymbol{\rho}) \mathrm{d}y + \int_0^1 N(f_u - \boldsymbol{\phi}_u \boldsymbol{\lambda} + \mathbf{S}_u \boldsymbol{\rho}) \mathrm{d}y \\ & + N\left[\int_0^1 (f_\pi - \boldsymbol{\phi}_\pi \boldsymbol{\lambda} + \mathbf{S}_\pi \boldsymbol{\rho}) \mathrm{d}y + (h_\pi + \boldsymbol{\omega}_\pi \boldsymbol{\sigma})_0 + (g_\pi + \boldsymbol{\psi}_\pi \boldsymbol{\mu})_1 \right] \\ & + N(-\boldsymbol{\lambda} + h_\chi + \boldsymbol{\omega}_\chi \boldsymbol{\sigma})_0 + N(\boldsymbol{\lambda} + g_\chi + \boldsymbol{\psi}_\chi \boldsymbol{\mu})_1. \end{aligned} \tag{12}$$

For the exact optimal solution, one must have

$$P \equiv 0, \quad Q \equiv 0. \tag{13a, b}$$

For an approximation to the optimal solution, the following relations are to be satisfied:

$$P \leqslant \varepsilon_1, \quad Q \leqslant \varepsilon_2. \tag{14a, b}$$

where $\varepsilon_1 (= 10^{-16})$ and $\varepsilon_2 (= 10^{-8})$ are preselected, small, positive numbers.

Sequential gradient-restoration algorithm

Over the past several years, a successful family of first-order algorithms for the solution of optimal control problems involving differential constraints, non-differential constraints and terminal constraints has been developed at Rice University by Miele and his associates.[13–17] They are called sequential gradient-restoration algorithms (SGRA) and have been designed for the solution of different classes of optimal control problems.

Sequential gradient-restoration algorithms involve a sequence of cycles, each with two-phases: the gradient phase and the restoration phase. In the gradient phase, the value of the augmented functional is decreased while avoiding excessive constraint violation; in the restoration phase, the constraint error is decreased while avoiding excessive change in the value of the functional. In a complete gradient-restoration cycle, the value of the functional is decreased while the constraints are satisfied to a predetermined accuracy. Hence, a succession of suboptimal solutions is obtained. The algorithmic details of SGRA can be found in References 13–17.

Application of the SGRA

The *SGRA* has been successfully applied to the study of the Benard problem by Lam and Bayazitoglu.[18] To show the applicability of the SGRA to the solution of the Orr–Sommerfeld equation, stability of a thin liquid film flowing down an inclined plane is solved.

The profiles of the stream function amplitude, $\Phi(y)$, and the wave velocity, c, are complex quantities:

$$\Phi = \Phi_r + i\Phi_i, \tag{15}$$

$$c = c_r + ic_i, \tag{16}$$

in which subscripts r and i denote the real and imaginary parts of the quantity. For computational purposes the solution is carried out in a real number system. Instead of solving the fourth-order equation (2), a system of fourth-order equations is solved. By substituting equations (15) and (16) into equations (2)–(7), and separately equating the real and imaginary parts of each equation to zero, we obtain the following modified eigenvalue problem.

Governing equation

$$\Phi_r'''' - 2\alpha^2\Phi_r'' + \alpha^4\Phi_r = -\alpha Re[(U - c_r)(\Phi_i'' - \alpha^2\Phi_i) - c_i(\Phi_r'' - \alpha^2\Phi_r) - U''\Phi_i], \tag{17}$$

$$\Phi_i'''' - 2\alpha^2\Phi_i'' + \alpha^4\Phi_i = \alpha Re[(U - c_r)(\Phi_r'' - \alpha^2\Phi_r) + c_i(\Phi_i'' - \alpha^2\Phi_i) - U''\Phi_r]; \tag{18}$$

Boundary conditions.

at the rigid surface, $y = 1$:

$$\Phi_r = 0, \tag{19}$$

$$\Phi_i = 0, \tag{20}$$

$$\Phi_r' = 0, \tag{21}$$

$$\Phi_i' = 0, \tag{22}$$

at the free surface, $y = 0$:

$$\Phi_r'' + \left[\alpha^2 - \frac{3(c_r - 1{\cdot}5)}{(c_r - 1{\cdot}5)^2 + c_i^2}\right]\Phi_r - \frac{3c_i}{(c_r - 1{\cdot}5)^2 + c_i^2}\Phi_i = 0, \tag{23}$$

$$\Phi_i'' + \left[\alpha^2 - \frac{3(c_r - 1{\cdot}5)}{(c_r - 1{\cdot}5)^2 + c_i^2}\right]\Phi_i + \frac{3c_i}{(c_r - 1{\cdot}5)^2 + c_i^2}\Phi_r = 0, \tag{24}$$

$$\frac{\alpha(3\cot\beta + \alpha^2 S_t Re)}{(c_r - 1{\cdot}5)^2 + c_i^2}[(c_r - 1{\cdot}5)\Phi_r + c_i\Phi_i] + \alpha Re(c_r - 1{\cdot}5)\Phi_r' - \alpha(c_i Re + 3\alpha)\Phi_i' + \Phi_i''' = 0, \tag{25}$$

$$\frac{\alpha(3\cot\beta + \alpha^2 S_t Re)}{(c_r - 1{\cdot}5)^2 + c_i^2}[(c_r - 1{\cdot}5)\Phi_i - c_i\Phi_r] + \alpha Re(c_r - 1{\cdot}5)\Phi_i' + \alpha(c_i Re + 3\alpha)\Phi_r' - \Phi_r''' = 0. \tag{26}$$

Now we are ready to demonstrate how the equations (17)–(26) for the stability problem can be recast into an optimal control problem. We employ scalar notation in the following development.

In particular, the symbol y denotes the non-dimensional spatial variable; the symbols $\chi_i(y)$, $i = 1, \ldots, m$, denote the components of the state (Φ_r and Φ_i and their derivatives) and the symbols $\pi_i = 1, \ldots, n$, denote the components of the parameter ($\pi_1^2 = \alpha, \pi_2^2 = Re, \pi_3^2 = c_r$). Once the conversion is completed, the resulting optimal control problem can be solved subject to the boundary conditions. We define eleven new variables:

$$\chi_m = \Phi_r^{(m-1)}, \qquad m = 1, 2, 3, 4$$
$$\chi_m = \Phi_i^{(m-5)}, \qquad m = 5, 6, 7, 8$$
$$\pi_1 = \sqrt{\alpha},$$
$$\pi_2 = \sqrt{Re},$$
$$\pi_3 = \sqrt{c_r}.$$

With these notations, the differential equations (17) and (18) may be written as a system of first-order differential equations. Thus, we recast the convective instability problem presented previously into an optimal control problem which can be stated as

minimize
$$I = \pi_2^2, \tag{27}$$

subject to the differential constraints

$$\chi_1' = \chi_2, \tag{28}$$
$$\chi_2' = \chi_3, \tag{29}$$
$$\chi_3' = \chi_4, \tag{30}$$
$$\chi_4' = 2\pi_1^4\chi_3 - \pi_1^8\chi_1 - \pi_1^2\pi_2^2[(U - \pi_3^2)(\chi_7 - \pi_1^4\chi_5) - c_i(\chi_3 - \pi_1^4\chi_1) - U''\chi_5], \tag{31}$$
$$\chi_5' = \chi_6, \tag{32}$$
$$\chi_6' = \chi_7, \tag{33}$$
$$\chi_7' = \chi_8, \tag{34}$$
$$\chi_8' = 2\pi_1^4\chi_7 - \pi_1^8\chi_5 + \pi_1^2\pi_2^2[(U - \pi_3^2)(\chi_3 - \pi_1^4\chi_1) + c_i(\chi_7 - \pi_1^4\chi_5) - U''\chi_1]; \tag{35}$$

and boundary conditions

at the rigid surface, $y = 1$:

$$\chi_1 = 0, \tag{36}$$
$$\chi_5 = 0, \tag{37}$$
$$\chi_2 = 0, \tag{38}$$
$$\chi_6 = 0; \tag{39}$$

at the free surface, $y = 0$

$$\chi_3 + \left[\pi_1^4 - \frac{3(\pi_3^2 - 1{\cdot}5)}{(\pi_3^2 - 1{\cdot}5)^2 + c_i^2}\right]\chi_1 - \frac{3c_i}{(\pi_3^2 - 1{\cdot}5)^2 + c_i^2}\chi_5 = 0, \tag{40}$$

$$\chi_7 + \left[\pi_1^4 - \frac{3(\pi_3^2 - 1{\cdot}5)}{(\pi_3^2 - 1{\cdot}5)^2 + c_i^2}\right]\chi_5 + \frac{3c_i}{(\pi_3^2 - 1{\cdot}5)^2 + c_i^2}\chi_1 = 0, \tag{41}$$

$$\frac{\pi_1^2(3\cot\beta + \pi_1^4\pi_2^2 S_t)}{(\pi_3^2 - 1{\cdot}5)^2 + c_i^2}[(\pi_3^2 - 1{\cdot}5)\chi_1 + c_i\chi_5] + \pi_1^2[\pi_2^2(\pi_3^2 - 1{\cdot}5)\chi_2 - (\pi_2^2 c_i + 3\pi_1^2)\chi_6] + \chi_8 = 0, \quad (42)$$

$$\frac{\pi_1^2(3\cot\beta + \pi_1^4\pi_2^2 S_t)}{(\pi_3^2 - 1{\cdot}5)^2 + c_i^2}[(\pi_3^2 - 1{\cdot}5)\chi_5 - c_i\chi_1] + \pi_1^2[\pi_2^2(\pi_3^2 - 1{\cdot}5)\chi_6 + (\pi_2^2 c_i + 3\pi_1^2)\chi_2] - \chi_4 = 0. \quad (43)$$

The system denoted by equations (27)–(43) constitutes an optimal control problem. In comparison with the Bolza problem outlined previously, it differs only by letting the functions f, g, the control (**u**) and the non-differential constraint (**S**) equal zero. The function h is simply π_2^2.

SOLUTION AND COMPUTATIONAL RESULTS

The SGRA was programmed in FORTRAN IV, and the numerical results were obtained in double-precision arithmetic. The interval of integration was divided into 100 steps. The differential systems were integrated using Hamming's modified predictor–corrector method with a special Runge–Kutta procedure to start the integration routine. The definite integrals, I, P, Q were computed using a modified Simpson's rule.

Once again, the symbol y denotes the non-dimensional spatial variable, the symbols $\chi_i(y)$, $i = 1, \ldots,$ denotes the components of the state (the real and imaginary part of the stream function amplitude, Φ_r and Φ_i and their derivatives) and π_1^2, π_2^2 and π_3^2 represent the wave number, α, the Reynolds number, Re, and the real part of the wave velocity, c_r, respectively.

Since all the differential constraints (28)–(35) and the boundary conditions (36)–(43) are homogeneous, we are free to impose a suitable normalization condition on the solution. The condition being used here is setting $\chi_3(1) = 1$.

The optimal control problem as designated by equations (27)–(43) was solved by specifying the values of c_i, β and S_t. Let us first examine the case for $\beta = 1°, S_t = 0$ and $c_i = 0$ for details. The following nominal functions are selected with discretion:

$$\chi_1 = (2y + 1)(1 - y)^2,$$
$$\chi_2 = 2(1 - y)^2 - 2(2y + 1)(1 - y),$$
$$\chi_3 = 2(2y + 1) - 8(1 - y),$$
$$\chi_4 = 12,$$
$$\chi_5 = -y^2(1 - y),$$
$$\chi_6 = -2y(1 - y) + y^2,$$
$$\chi_7 = 6y - 2,$$
$$\chi_8 = 6,$$
$$\pi_1 = 0{\cdot}5,$$
$$\pi_2 = 8,$$
$$\pi_3 = 2,$$

which are not consistent with the boundary conditions (36)–(43) and violate the constraints (28)–(35). Since these nominal functions do not constitute a feasible solution, the algorithm starts with a restoration phase. This restoration phase included $N_r = 13$ iterations and leads to new nominal functions (labelled $N_c = 1$) consistent with the equations (28)–(43) within the preselected accuracy

(14a). Next, the algorithm was employed cyclically until a solution consistent with inequalities (14) was found. This situations arose for $N_c = 13$, that is at the end of 13 cycles of the algorithm.

The numerical results are presented in Table I. It shows the number of gradient iterations per cycle. N_g, and the number of restorative iterations per cycle, N_r, versus the cycle number, N_c. Clearly, the total number of iterations for convergence is

$$N_t = \sum N_g + \sum N_r = 13 + 43 = 56.$$

Table I also shows the functional $I(=\pi_2^2 = Re)$, the constraint error P and the error in the optimality conditions Q versus the cycle number N_c.

Results for the neutral curve are given in Table II and Figures 1 and 2. The eigenfunction for an inclination angle of $\beta = 1°(Re_c = 47{\cdot}74168, \alpha = 0, c_r = 3, S_t = 0)$ is given in Figure 3.

Critical Reynolds numbers, wave numbers and the real parts of the wave velocity for overstability are determined for various non-dimensional surface tensions, S_t. The results are shown in Table III. One can see that surface tension is the stabilizing factor.

The results obtained from the present study are compared with the results obtained by different methods in Table IV for various angles of inclination. The reduction of the angle inclination is also

Table I. Convergence history for $\beta = 1°$, $c_i = 0$ and $S_t = 0$

N_c	N_g	N_r	P	Q	I
0	—	—	$0{\cdot}502 \times 10^{5}$	—	64.0
1	1	13	$0{\cdot}124 \times 10^{-18}$	$0{\cdot}139 \times 10^{2}$	49·2
2	1	3	$0{\cdot}199 \times 10^{-28}$	$0{\cdot}421 \times 10^{1}$	48·5
3	1	3	$0{\cdot}249 \times 10^{-28}$	0·636	48·0
4	1	2	$0{\cdot}303 \times 10^{-17}$	0·131	47·8
5	1	3	$0{\cdot}471 \times 10^{-27}$	$0{\cdot}298 \times 10^{-1}$	47·8
6	1	3	$0{\cdot}454 \times 10^{-27}$	$0{\cdot}817 \times 10^{-2}$	47·7
7	1	3	$0{\cdot}183 \times 10^{-28}$	$0{\cdot}250 \times 10^{-2}$	47·7
8	1	3	$0{\cdot}496 \times 10^{-30}$	$0{\cdot}792 \times 10^{-3}$	47·7
9	1	2	$0{\cdot}919 \times 10^{-16}$	$0{\cdot}249 \times 10^{-3}$	47·7
10	1	2	$0{\cdot}480 \times 10^{-17}$	$0{\cdot}732 \times 10^{-4}$	47·7
11	1	2	$0{\cdot}106 \times 10^{-18}$	$0{\cdot}170 \times 10^{-4}$	47·7
12	1	2	$0{\cdot}231 \times 10^{-21}$	$0{\cdot}169 \times 10^{-5}$	47·7
13	1	2	$0{\cdot}193 \times 10^{-27}$	$0{\cdot}468 \times 10^{-8}$	47·7

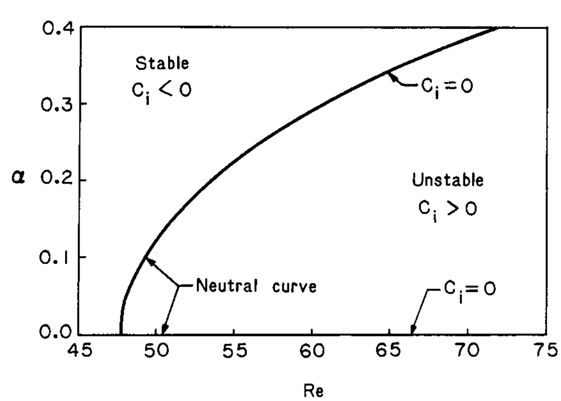

Figure 1. The neutral stability curve for the wave number as a function of the Reynolds number for $\beta = 1°$ and $S_t = 0$

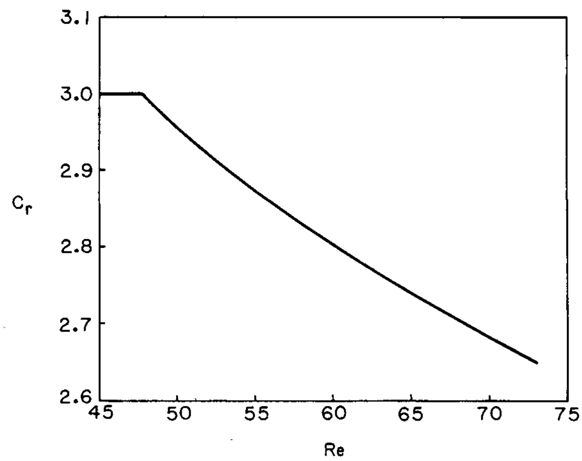

Figure 2. The neutral stability curve for the wave velocity c_r as a function of the Reynolds number for $\beta = 1^\circ$ and $S_t = 0$

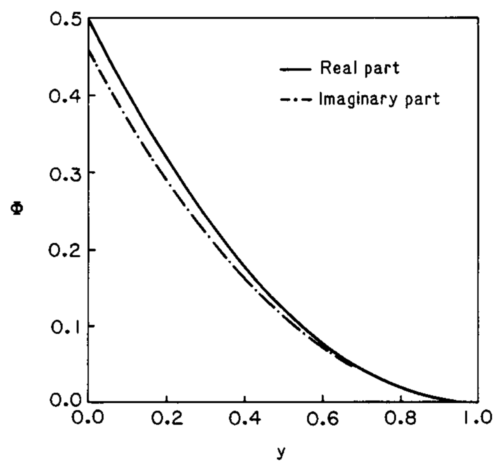

Figure 3. The eigenfunction $\Phi(y)$ for the case $\beta = 1^\circ$, $S_t = 0$, $Re = 47{\cdot}74168$, $\alpha = 0{\cdot}000296$ and $c_r = 3{\cdot}0$. The normalized condition is $\Phi_r''(1) = 1{\cdot}0$

Table II. Values of Re, α and c_r for points on the neutral stability curve for $\beta = 1^0$ and $S_t = 0$.

Re	α	c_r
49·20626	$0{\cdot}9819518 \times 10^{-1}$	2·973069
48·53859	$0{\cdot}7163170 \times 10^{-1}$	2·985241
48·00664	$0{\cdot}4086090 \times 10^{-1}$	2·995066
47·82530	$0{\cdot}2286410 \times 10^{-1}$	2·998439
47·76519	$0{\cdot}1211373 \times 10^{-1}$	2·999561
47·74885	$0{\cdot}6698733 \times 10^{-2}$	2·999866
47·74403	$0{\cdot}3857569 \times 10^{-2}$	2·999956
47·74248	$0{\cdot}2279302 \times 10^{-2}$	2·999983
47·74194	$0{\cdot}1367853 \times 10^{-2}$	2·999994
47·74176	$0{\cdot}8343063 \times 10^{-3}$	2·999997
47·74170	$0{\cdot}5225243 \times 10^{-3}$	3·0
47·74168	$0{\cdot}3549682 \times 10^{-3}$	3·0
47·74168	$0{\cdot}2955597 \times 10^{-3}$	3·0

Table III. Critical Re, α and c_r for the case $\beta = 90°$ for various S_t and c_i

	$c_i = 0{\cdot}05$			$c_i = 0{\cdot}1$			$c_i = 0{\cdot}2$		
S_t	Re	α	c_r	Re	α	c_r	Re	α	c_r
0	0·214	0·338	2·713	0·437	0·335	2·706	0·952	0·322	2·677
500	1·309	0·048	2·989	2·692	0·047	2·976	6·067	0·044	2·922
1000	1·836	0·034	2·992	3·770	0·034	2·979	8·518	0·031	2·925

Table IV. Comparison of the critical Reynolds numbers, Re_c for $S_t = 0$

90°	45°	1°	30′	10′	Method of solution
0	0·833	47·742	95·491	286·478	Power series expansion of y; Reference 5 Power series expansion of α; Reference 6 Asymptotic expansion; Reference 10
0	—	48·0	—	286·667	Asymptotic series solution and Frobenius method; Reference 11
0	—	48·0	—	—	Numerical integration with orthonormalization technique; Reference 12
0	0·833	47·742	95·491	286·467	Present study

the stabilizing factor. This comparison shows that the sequential gradient-restoration algorithm is an excellent numerical method for the solution of Orr–Sommerfeld problems.

CONCLUDING REMARKS

The sequential gradient-restoration algorithm (SGRA) is presented for the solution of the Orr–Sommerfeld equation. The agreement of the results of this study with the existing solutions is excellent, as shown in Table IV. The present method is rapid and highly accurate. Once the computer code of the procedure is completed, it requires only to change the governing equation and boundary conditions for different problems. It is possible to compute the neutral stability curves, eigenvalues and eigenfunctions simultaneously.

The method requires a choice of trial functions at the beginning of the iterative process similar to the Galerkin method, but the trial functions used in the SGRA need not satisfy the boundary conditions or the differential constraints. Provided that the initial guess in the iteration procedure is chosen with discretion, convergence to a solution is quite rapid, and the accuracy of the solution is limited only by the error allowed in calculating the performance indexes. Small allowable error gives infinite-order accuracy in the approximate solution of the linearized stability problem. On the contrary, the Galerkin method requires an extensive search for suitable trial functions, and higher order approximations are needed to obtain accurate results. The SGRA also has an advantage over the schemes by power series expansion or asymptotic representation: it does not require an extensive mathematical investigation.

The present numerical method is not free from difficulties. If the slopes of the actual eigenfunctions in the stability problem change rapidly (i.e. spline) over the range of integration, as in the case of the plane Poiseuille probelm, a close estimate of the eigenfunctions for the iterative procedure might be required to converge to the exact values within a reasonable number of iterations during the process.

In summary, the sequential gradient-restoration algorithm described in this work provides a powerful tool for finding solutions to the Orr–Sommerfeld equation. It is a straightforward matter to use the SGRA to study hydrodynamic stability problems. The method should be useful in the investigation of the stability of other flows. The technique is applicable to linear as well as non-linear differential equations without modification. It gives convenient, accurate, and efficient approximations to the solution of hydrodynamic stability problems.

NOMENCLATURE

c	wave velocity
d	film thickness
f	function appearing in Equation (8)
g	function appearing in Equation (8)
h	function appearing in Equation (8)
i	$\sqrt{-1}$
I	functional defined by Equation (8)
N_c	number of cycles
N_g	total number of iterations in the gradient phase
N_r	total number of iterations in the restoration phase
N_t	total number of iterations for the problem
P	performance index for the constraint error
Q	performance index for the optimality condtions
Re	Reynolds number
$\mathbf{S}$	non-differential constraint, equation (9b)
S_t	dimensionless surface tension
$\mathbf{u}$	control, equation (8)
U	dimensionless velocity of the primary flow
x, y	dimensionless co-ordinates
α	wave number
β	angle of inclination
$\boldsymbol{\lambda}$	Lagrange multiplier associated with $\boldsymbol{\phi}$
$\boldsymbol{\mu}$	Lagrange multiplier associated with $\boldsymbol{\psi}$
$\boldsymbol{\pi}$	parameter appearing in equation (8)
$\boldsymbol{\rho}$	Lagrange multiplier associated with $\mathbf{S}$
$\boldsymbol{\sigma}$	Lagrange multiplier associated with $\boldsymbol{\omega}$
$\boldsymbol{\phi}$	defined by equation (9a)
$\boldsymbol{\Phi}$	stream function amplitude
$\boldsymbol{\chi}$	dummy variable for $\boldsymbol{\Phi}$
$\boldsymbol{\psi}$	final condition, equation (9d)
$\boldsymbol{\omega}$	initial condition, equation (9c)

Subscripts

c	critical quantity
i	imaginary part
r	real part

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